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Bohr-compactifications are cocompactifications

by

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Let G be a topological group. A topological group G^* is called a cogroup of G if there exists a map ϕ of G onto G^* which satisfies the following two conditions:

(i) ϕ is an isomorphism of the abstract group G onto the abstract group G^*;

(ii) ϕ is a compression map of the topological space G onto the topological space G^*; in other words, G^* is a cospace of G.

(For the concepts cospace, compression map and - below - closed base see [3]).

A cocompactification of G is a compact group G' which contains a cogroup of G as a dense subgroup.

We will prove that every locally compact abelian group (abbreviated to: LCA-group) has a cocompactification. The BOHR-compactification bG of an LCA-group G (cf. [1] or [2]) is an example of a cocompactification of G; conversely every cocompactification is topologically isomorphic to a generalized BOHR-compactification of G.

Proposition 1. Let G be a locally compact group, and let ϕ be a continuous isomorphism of G in a compact group H. The closure of $\phi(G)$ in H is a cocompactification of G.

Proof.

The collection Γ of all compact subsets of G is a closed base in G. As $\phi(C)$ is compact, hence closed, in ϕG whenever $C \in \Gamma$, the assertion follows from [3], §1.2, proposition 3.

Corollary. Every LCA-group G has a cocompactification. The BOHR-compactification bG of G is a cocompactification of G.

Proof.

It is known that there exists a continuous isomorphism of G onto a dense subgroup of bG (see e.g. [2], theorem 1.8.2).

In general, the BOHR-compactification bG is not the only cocompactification of the LCA-group G.

Example. Let Z denote the additive group of integers and T the circle group. The BOHR-compactification bZ of Z is the charactergroup of the discrete group T^d (the group T with the discrete topology). This group bZ is very large (it has $2^{\mathfrak{c}}$ points, where $\mathfrak{c}$ is the cardinal number of the continuum; also its weight is $2^{\mathfrak{c}}$). However, every monothetic group is a cocompactification of Z (and every cocompactification of Z is a monothetic group); in particular, T itself is a cocompactification of Z.

A characterization of all cocompactifications of an LCA-group G can be obtained by means of the generalized BOHR-compactifications of G (cf. [1] §26). Let Σ be a collection of continuous characters of G. A map Φ_Σ of G into T^Σ is defined as follows:

$$\Phi_\Sigma(x) = \big(\chi(x)\big)_{\chi \in \Sigma} \quad , \quad \text{for all } x \in G.$$

Let $b_\Sigma G$ be the closure of $\Phi_\Sigma(G)$ in T^Σ. It is easily seen that $b_\Sigma G$ is a subgroup of T^Σ, and that Φ_Σ is a continuous homomorphism of G into $b_\Sigma G$. Moreover, Φ_Σ is an isomorphism if and only if Σ separates the points of G. If Σ consists of all continuous characters on G, then $b_\Sigma G = bG$.

From [1], theorem 26.13, we now conclude:

Proposition 2. Let G be an LCA-group. If Σ is a set of continuous characters of G which separates the points of G, then $b_\Sigma G$ is a cocompactification of G, and Φ_Σ is a continuous isomorphism of G onto a dense subgroup of $b_\Sigma G$. Conversely every cocompactification of G is obtained in this way up to a topological isomorphism.

Returning to our example above, the cocompactification T of Z can be obtained as a $b_\Sigma Z$, where Σ consists of a single non-trivial character on Z. In other words, $\Sigma = \{\chi\}$ with $\chi(n) = e^{2\pi i n\theta}$, θ a fixed irrational number, for all $n \in \Sigma$.

References:

[1] E. HEWITT and K.A. ROSS, Abstract harmonic analysis I. Berlin, 1963.

[2] W. RUDIN, Fourier analysis on groups. New York, 1962.

[3] Syllabus of a colloquium on cotopology, Mathematical Centre, Amsterdam, 1964- . . .

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