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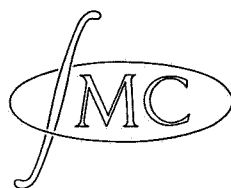
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Bohr-compactifications are cocompactifications

by

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## Bohr-compactifications are cocompactifications

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Let  $G$  be a topological group. A topological group  $G^*$  is called a cogroup of  $G$  if there exists a map  $\phi$  of  $G$  onto  $G^*$  which satisfies the following two conditions:

- (i)  $\phi$  is an isomorphism of the abstract group  $G$  onto the abstract group  $G^*$ ;
  - (ii)  $\phi$  is a compression map of the topological space  $G$  onto the topological space  $G^*$ ; in other words,  $G^*$  is a cospace of  $G$ .
- (For the concepts cospace, compression map and - below - closed base see [3]).

A cocompactification of  $G$  is a compact group  $G'$  which contains a cogroup of  $G$  as a dense subgroup.

We will prove that every locally compact abelian group (abbreviated to: LCA-group) has a cocompactification. The BOHR-compactification  $bG$  of an LCA-group  $G$  (cf. [1] or [2]) is an example of a cocompactification of  $G$ ; conversely every cocompactification is topologically isomorphic to a generalized BOHR-compactification of  $G$ .

Proposition 1. Let  $G$  be a locally compact group, and let  $\phi$  be a continuous isomorphism of  $G$  in a compact group  $H$ . The closure of  $\phi(G)$  in  $H$  is a cocompactification of  $G$ .

Proof.

The collection  $\Gamma$  of all compact subsets of  $G$  is a closed base in  $G$ . As  $\phi(C)$  is compact, hence closed, in  $\phi G$  whenever  $C \in \Gamma$ , the assertion follows from [3], §1.2, proposition 3.

Corollary. Every LCA-group  $G$  has a cocompactification. The BOHR-compactification  $bG$  of  $G$  is a cocompactification of  $G$ .

Proof.

It is known that there exists a continuous isomorphism of  $G$  onto

a dense subgroup of  $bG$  (see e.g. [2], theorem 1.8.2).

In general, the BOHR-compactification  $bG$  is not the only cocompactification of the LCA-group  $G$ .

Example. Let  $Z$  denote the additive group of integers and  $T$  the circle group. The BOHR-compactification  $bZ$  of  $Z$  is the charactergroup of the discrete group  $T^d$  (the group  $T$  with the discrete topology). This group  $bZ$  is very large (it has  $2^{\mathfrak{c}}$  points, where  $\mathfrak{c}$  is the cardinal number of the continuum; also its weight is  $2^{\mathfrak{c}}$ ). However, every monothetic group is a cocompactification of  $Z$  (and every cocompactification of  $Z$  is a monothetic group); in particular,  $T$  itself is a cocompactification of  $Z$ .

A characterization of all cocompactifications of an LCA-group  $G$  can be obtained by means of the generalized BOHR-compactifications of  $G$  (cf. [1] §26). Let  $\Sigma$  be a collection of continuous characters of  $G$ . A map  $\phi_\Sigma$  of  $G$  into  $T^\Sigma$  is defined as follows:

$$\phi_\Sigma(x) = (\chi(x))_{\chi \in \Sigma}, \quad \text{for all } x \in G.$$

Let  $b_\Sigma G$  be the closure of  $\phi_\Sigma(G)$  in  $T^\Sigma$ . It is easily seen that  $b_\Sigma G$  is a subgroup of  $T^\Sigma$ , and that  $\phi_\Sigma$  is a continuous homomorphism of  $G$  into  $b_\Sigma G$ . Moreover,  $\phi_\Sigma$  is an isomorphism if and only if  $\Sigma$  separates the points of  $G$ . If  $\Sigma$  consists of all continuous characters on  $G$ , then  $b_\Sigma G = bG$ .

From [1], theorem 26.13, we now conclude:

Proposition 2. Let  $G$  be an LCA-group. If  $\Sigma$  is a set of continuous characters of  $G$  which separates the points of  $G$ , then  $b_\Sigma G$  is a cocompactification of  $G$ , and  $\phi_\Sigma$  is a continuous isomorphism of  $G$  onto a dense subgroup of  $b_\Sigma G$ . Conversely every cocompactification of  $G$  is obtained in this way up to a topological isomorphism.

Returning to our example above, the cocompactification  $T$  of  $Z$  can be obtained as a  $b_{\Sigma}Z$ , where  $\Sigma$  consists of a single non-trivial character on  $Z$ . In other words,  $\Sigma = \{\chi\}$  with  $\chi(n) = e^{2\pi i n \theta}$ ,  $\theta$  a fixed irrational number, for all  $n \in Z$ .

References:

- [1] E. HEWITT and K.A. ROSS, Abstract harmonic analysis I. Berlin, 1963.
- [2] W. RUDIN, Fourier analysis on groups. New York, 1962.
- [3] Syllabus of a colloquium on cotopology, Mathematical Centre, Amsterdam, 1964- . . .